

Almost-free finite covers

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Abstract

Let W be a first-order structure and ρ be an $\text{Aut}(W)$ -congruence on W . In this paper we define the *almost-free* finite covers of W with respect to ρ , and we show how to construct them. These are a generalization of free finite covers.

A consequence of a result of [5] is that any finite cover of W with binding groups all equal to a simple non-abelian permutation group is almost-free with respect to some ρ on W . Our main result gives a description (up to isomorphism) in terms of the $\text{Aut}(W)$ -congruences on W of the kernels of principal finite covers of W with bindings groups equal at any point to a simple non-abelian regular permutation group G . Then we analyze almost-free finite covers of $\Omega^{(n)}$, the set of ordered n -tuples of distinct elements from a countable set Ω , regarded as a structure with $\text{Aut}(\Omega^{(n)}) = \text{Sym}(\Omega)$ and we show a result of biinterpretability.

The material here presented addresses a problem which arises in the context of classification of totally categorical structures.

1 Introduction

Given an countable set W , consider the natural action of the symmetric group $\text{Sym}(W)$ on W . This action yields a topology on $\text{Sym}(W)$ in which pointwise stabilizers of finite sets give a base of open neighborhoods of the identity. Let Υ be a closed subgroup of $\text{Sym}(W)$ that acts transitively on W and G a finite group acting on a finite set Δ . Consider the projection $\pi : \Delta \times W \rightarrow W$ given by $\pi(\delta, w) = w$. We denote by G^W the set of all functions from W to G . Let \mathcal{F} be the set of closed subgroups of $\text{Sym}(\Delta \times W)$ which preserve the partition of $\Delta \times W$ given by the fibres of π . Every $F \in \mathcal{F}$ determines naturally an induced map $\mu_F : F \rightarrow \text{Sym}(W)$. Additionally we require that, for all $F \in \mathcal{F}$, $\mu_F(F) = \Upsilon$ and the permutation groups induced respectively by F and $\ker \mu_F$ on $\pi^{-1}(w)$, for all $w \in W$, are both equal to G . Let $\mathcal{K} = \{\ker \mu_F, F \in \mathcal{F}\}$. In this paper we will deal with the following

Problem: Given G and Υ , find a description of the elements belonging to \mathcal{K} .

This problem, which is here formulated in terms of infinite permutation groups, is motivated by questions arising in model theory concerning finite covers (see [6]).

Definition 1 Let C and W be two first-order structures. A finite to-one surjection $\pi : C \rightarrow W$ is a finite cover if its fibres form an $\text{Aut}(C)$ -invariant partition of C , and the induced map $\mu : \text{Aut}(C) \rightarrow \text{Sym}(W)$, defined by $\mu(g)(w) = \pi(g\pi^{-1}(w))$, for all $g \in \text{Aut}(C)$ and for all $w \in W$, has image $\text{Aut}(W)$.

We shall refer to the kernel of μ as the kernel of the finite cover π . If $\pi : C \rightarrow W$ is a finite cover, the fibre group $F(w)$ at $w \in W$ is the permutation group induced by $\text{Aut}(C)$ on $\pi^{-1}(w)$. The binding group $B(w)$ at $w \in W$ is the permutation group induced by the kernel on $\pi^{-1}(w)$.

Using the terminology of finite covers, the problem above can be stated in the following equivalent version: given a finite group G and a first-order structure W with automorphism group Υ , describe the kernels of the finite covers of W with $F(w) = B(w) = G$ at any point, which have $\Delta \times W$ as domain of the covering structures and are the projection on the second coordinate.

A more detailed commentary on finite covers and this problem is given in the last section. However, we avoid the model-theoretic methods using rather infinite permutation groups techniques.

In [2] Ahlbrandt and Ziegler described the subgroups $K \in \mathcal{K}$, when G is an abelian permutation group. In this case G^W , the group of the function from W to G , is a Υ -module with $f^v(w) = f(v^{-1}w)$, where $v \in \Upsilon$ and $f \in G^W$ and the kernels in \mathcal{K} are profinite Υ -modules. They proved that \mathcal{K} is exactly the set of closed Υ -submodules of G^W .

In this paper, we deal with the case when G is a simple *non abelian* regular permutation group. Under this hypothesis our main result, which is stated and proved in Section 3), gives a description of the elements of \mathcal{K} in terms of the Υ -congruences on W . A key ingredient in the proof is a result of Evans and Hrushovski ([5], Lemma 5.7).

Previous results are the following. In [10], Ziegler described the groups $K \in \mathcal{K}$ in the case when W is a countable set Ω and $\Upsilon = \text{Sym}(\Omega)$ (the disintegrated case), for any group G . Increasing the complexity of the set W , it seems not possible to give a general description of the groups $K \in \mathcal{K}$ not depending on the group G . For example, if W is the set on n -subsets from a countable set Ω , $\Upsilon = \text{Sym}(\Omega)$ and G is a cyclic group of order a prime p , then the groups $K \in \mathcal{K}$ are an intersection of kernels of certain Υ -homomorphism, as it is described in [7]. While if G is a simple non abelian group, then $\mathcal{K} = \{G, G^W\}$ (see corollary 6). In Section 4 we analyze the special case in which given a countable set Ω , W is defined as the subset of the n -fold cartesian product $\Omega^{(n)}$ whose elements are n -tuples with pairwise distinct entries. Defining Υ as $\text{Sym}(\Omega)$, in Proposition 16 and 17 we give an explicit description of the equivalence classes of the $\text{Sym}(\Omega)$ -congruences on $\Omega^{(n)}$. In these Propositions we see that the blocks for $\text{Sym}(\Omega)$ in $\Omega^{(n)}$ can be either of finite or of infinite cardinalities. Proposition 23 shows that if $\pi : C \rightarrow \Omega^{(n)}$ is a cover of $\Omega^{(n)}$ with $\text{Aut}(C)$ in \mathcal{F} and G equal to a simple-non abelian finite group such that the kernel of π determines a $\text{Sym}(\Omega)$ -congruence on $\Omega^{(n)}$ (in the sense of Lemma 5) with classes of finite cardinality, then, for every $m \in \mathbb{N}$ greater than n , there exists a finite cover $\pi' : C' \rightarrow \Omega^{(m)}$ bi-interpretable with π with binding groups and fibre groups both equal to G at any point and kernel that determines a $\text{Sym}(\Omega)$ -congruence on $\Omega^{(m)}$ with classes of infinite cardinality.

In section 5.3 we define the almost-free finite covers. A posteriori we see that the results of sections 3 and 4 concern examples of almost-free finite covers with binding groups equal to the fibre groups at any point. Let W be a transitive structure, ρ be an $\text{Aut}(W)$ -congruence on W and $[w_0]$ be a congruence class. An almost-free finite cover π of W w.r.t ρ is a finite cover whose permutation group induced by its kernel on the union of the fibres of π over $[w_0]$ is isomorphic to the binding group at w_0 , while the permutation group induced on the fibres over two elements not in the same congruence class is the direct product of the two respective binding groups. This definition generalizes the definition of free finite cover. More in detail a free finite cover of W is an almost-free finite cover of W with respect to the equality. In Proposition 26 we show how to construct an almost-free finite cover. The proof uses Lemma 2.1.2 of [6].

2 General results

Definition 2 *A pregeometry on a set X is a relation between elements $x \in X$ and finite subsets $X_0 \subset X$, called dependence, which satisfies:*

- *Reflexivity : x is dependent on $\{x\}$;*
- *Extension: x depends on X_0 and $X_0 \subseteq X_1$ implies x depends on X_1 ;*
- *Transitivity: x is dependent on X_0 and every $y \in X_0$ is dependent on X_1 implies x is dependent on X_1 ;*
- *Symmetry: x is dependent on $X_0 \cup \{y\}$ but not on X_0 , implies y is dependent on $X_0 \cup \{x\}$.*

Remark 3 *A classical example of a pregeometry is a vector space with linear dependency.*

If Ω is any set then there is a natural topology on $\text{Sym}(\Omega)$ which makes it into a topological group. The open sets are unions of cosets of pointwise stabilizers of finite subsets of Ω . We then make any permutation group P on Ω into a topological group by giving it the relative topology. If Ω is countable the topology is metrisable.

From now on W stands for a countable set, Υ for a closed subgroup of $\text{Sym}(W)$ that acts transitively on W and G for a finite group acting on a finite set Δ . Consider the projection $\pi : \Delta \times W \rightarrow W$ given by $\pi(\delta, w) = w$. We denote by G^W the set of all functions from W to G . Let \mathcal{F} be the set of closed subgroups of $\text{Sym}(\Delta \times W)$ which preserve the partition of $\Delta \times W$ given by the fibres of π . Every $F \in \mathcal{F}$ determines naturally an induced map $\mu_F : F \rightarrow \text{Sym}(W)$. Additionally we require that, for all $F \in \mathcal{F}$, $\mu_F(F) = \Upsilon$ and the permutation groups induced respectively by F and $\ker \mu_F$ on $\pi^{-1}(w)$, for all $w \in W$, are both equal to G . We notice that the wreath product $GW r_W \Upsilon$ in its imprimitive action on $\Delta \times W$ belongs to \mathcal{F} .

It is easy to see that, with the above topology, G^W is a compact subgroup of $\text{Sym}(\Delta \times W)$ and $\ker \mu_F$ are closed subgroups of G^W and that μ_F are continuous and open maps (Lemma 1.4.2, [6]). We introduce now a notion of isomorphism among the elements of \mathcal{F} . We say that F_1 and F_2 are *isomorphic* if there exists a bijection $\phi : \Delta \times W \rightarrow \Delta \times W$ which sends $\phi(\pi^{-1}(w)) = \pi^{-1}(w)$,

for all $w \in W$ and such that the induced map $f_\phi : \text{Sym}(\Delta \times W) \rightarrow \text{Sym}(\Delta \times W)$ sends F_1 to F_2 . Let $\mathcal{K} = \{\ker \mu_F, F \in \mathcal{F}\}$. We now introduce the following equivalence relation R on \mathcal{K} : $\ker \mu_{F_1} R \ker \mu_{F_2}$ if and only if F_1 is isomorphic to F_2 and we denote the R -equivalence class of an arbitrary $K \in \mathcal{K}$ by $[K]$. (We shall say that $\ker \mu_{F_1}$ is *isomorphic* to $\ker \mu_{F_2}$ if F_1 is isomorphic to F_2 .)

Take $K \in \mathcal{K}$ and $w_1, \dots, w_k \in W$. We define

$$K(w_1, \dots, w_k) = \{f|_{\{w_1, \dots, w_k\}} \mid f \in K\}$$

and, for simplicity, we shall refer to $K(w_1, \dots, w_k)$ as K restricted to w_1, \dots, w_k .

Definition 4 Suppose w_1, \dots, w_k, w belong to W . We say that w depends on w_1, \dots, w_k and write $w \in \text{cl}(w_1, \dots, w_k)$, if

$$K(w, w_1, \dots, w_k) \cong G$$

Lemma 5 ([5], Lemma 5.7) Let $K \in \mathcal{K}$ and $w_1, \dots, w_k, w \in W$. Then (W, cl) is a Υ -invariant pregeometry. If G is a simple non-abelian finite group, then (W, cl) reduces to an equivalence relation.

The lemma states that, if G is non-abelian and w depends on w_1, \dots, w_k , then there is an $i \in \{1, \dots, k\}$ such that w depends on w_i and (W, cl) is a Υ -congruence.

Corollary 6 If Υ acts primitively on W and G is a simple non-abelian finite group, then $\mathcal{K} = \{G, G^W\}$.

Here there are some results on topological groups that will be useful in the next section.

Lemma 7 Let G be a permutation group on an infinite set Ω with the usual topology. A subgroup H of G is open in this topology if and only if H contains $G_{(\Gamma)}$ for some finite Γ .

Take a typical basic open set of $\text{Aut}(C_1) = G_1$:

$$(G_1)_{(F)} = \{g_1 \in G_1 : g_1(f) = f, \forall f \in F\}$$

for some finite $F \subset C_1$. Let $a_i \in \Omega$, for $i \in \{1, \dots, n\}$, and $\lambda \in \Lambda$. The preimage of $(G_1)_{(F)}$ under Φ is $\Phi^{-1}((G_1)_{(F)}) = \{g \in \text{Aut}(C) : g(f') = f', \forall f' \in F'\}$, where $F' = \{(\lambda, a_1, \dots, a_m) : (\lambda, a_1, \dots, a_n) \in F\}$ is a finite set. Hence, $\Phi^{-1}((G_1)_{(F)}) = \text{Aut}(C/F')$, which is open and Φ is continuous.

Proposition 8 Let G be a topological group and let H be a subgroup of G . Then, if G is compact and H closed, H is compact.

For a proof of the previous proposition see for instance [8] Chapter 2, paragraph 8,10.

Proposition 9 Let G be a topological group. Suppose G is metrisable. Let A be a compact subgroup of G and B a closed subgroup of G . Then AB and BA are closed sets.

Proof. It is sufficient to show that AB is closed. Let $\{c_n\}_{n \in \mathbb{N}}$ be a sequence of elements of AB which converges to c . We have $c_n = a_n b_n$, where $a_n \in A$ and $b_n \in B$. Since A is compact, we can select from the sequence $\{a_n\}_{n \in \mathbb{N}}$ a subsequence $\{a_{n_k}\}$ which converges to an element $a \in A$. We conclude from the convergence of the sequences $\{c_{n_k}\}$ and $\{a_{n_k}\}$ that the sequence $\{b_{n_k}\}$ converges to the element $a^{-1}c$, which belongs to B , since B is closed. Hence $c = a(a^{-1}c) \in AB$ and the closure of the set AB is established. ■

3 Main Theorem

We will denote by \mathcal{C} the set of all Υ -congruences on W .

Definition 10 Let $\rho \in \mathcal{C}$. We define the subgroup of G^W

$$K_\rho = \{f : W \rightarrow G : f \text{ constant on } Y, \forall Y \in W/\rho\}.$$

Theorem 11 Let G be a simple non-abelian finite permutation group acting regularly on a finite set Λ . Then there exists a bijection Ψ between \mathcal{C} and \mathcal{K}/R given by $\Psi(\rho) = [K_\rho]$. The inverse mapping Φ of Ψ is given by $\Phi([K]) = \rho_K$, where ρ_K is defined by:

$$w_i \rho_K w_j \Leftrightarrow K(w_i, w_j) \cong G.$$

Proof. We first show that Ψ maps \mathcal{C} into \mathcal{K}/R .

Let $\rho \in \mathcal{C}$. Then K_ρ is a subgroup of G^W . First of all we embed K_ρ into $G^W \rtimes \Upsilon$ in the natural way:

$$\begin{array}{ccc} K_\rho & \hookrightarrow & G^W \rtimes \Upsilon \\ f & \mapsto & (f, 1) \end{array}$$

and then we notice that K_ρ is normalized by Υ . Indeed, given $\sigma \in \Upsilon$, we have that

$$(\sigma(f), 1)(\lambda, w) := (1, \sigma)(f, 1)(1, \sigma^{-1})(\lambda, w) = (f(\sigma^{-1}w)\lambda, w).$$

Since $f \in K_\rho$, for every $w_i \in [w_j]_\rho$ in W we have $f(w_i) = f(w_j)$, but, since ρ is a Υ -congruence on W , we have $f(\sigma^{-1}w_i) = f(\sigma^{-1}w_j)$, for every $w_i \in [w_j]_\rho$ and so $(\sigma(f), 1) \in K_\rho$.

Since K_ρ is normalized by Υ , we can consider the group:

$$H := K_\rho \rtimes \Upsilon.$$

This is a subgroup of $G^W \rtimes \Upsilon$ and if $\mu : GWr_W \Upsilon \rightarrow \Upsilon$ is the map defined by $\mu(f, \gamma) = \gamma$, we then have that $\mu(H) = \Upsilon$ and $\ker \mu = K_\rho$. In order to prove that K_ρ is an element of \mathcal{K} it is sufficient to show that H is a closed subgroup of $G^W \rtimes \Upsilon$. Indeed, $G^W \rtimes \Upsilon$ is closed in $\text{Sym}(\Delta \times W)$.

The first step is to prove that K_ρ is closed.

The finite group G has the discrete topology, while G^W has the product topology. An element $f \in G^W$ is a function from W to G . The w -projection map is the map $\pi_w : G^W \rightarrow G$ such that $\pi_w(f) = f(w)$. A basis for the product topology on G^W is the family of all finite intersections of $\pi_w^{-1}(U)$, where U is an open subset of G . In this topology the maps π_w are continuous. Hence, a member of this basis is of the form

$$\bigcap \{\pi_w^{-1}(U_w) : w \in F\}$$

where F is a finite subset of W .

Let $[w]_\rho$ be a ρ -class and g an element of the simple finite group G . By the continuity of π_w , $\pi_w^{-1}(g)$ is a closed subset of G^W . Let

$$M_{[w]_\rho}(g) := \bigcap_{v \in [w]_\rho} \pi_v^{-1}(g).$$

Then $M_{[w]_\rho}(g)$ is a closed set in G^W . We consider next

$$\bigcup_{g \in G} M_{[w]_\rho}(g)$$

and this is still a closed subset of K_0 . Then, if Σ is the set of all the equivalence classes of ρ ,

$$K_\rho = \bigcap_{[w]_\rho \in \Sigma} \bigcup_{g \in G} M_{[w]_\rho}(g)$$

and so K_ρ is closed in K_0 .

Since K_ρ is a closed subgroup of the compact group G^W , K_ρ is compact by Proposition 8. By Proposition 9, $H = K_\rho \rtimes \Upsilon$ is closed. Thus, we have shown that Ψ maps \mathcal{C} to \mathcal{K}/R .

It's easy to see that the map Φ is well defined. Finally, Lemma 5 shows that $\Phi([K]) \in \mathcal{C}$.

In order to prove that Ψ is a bijection, we show that $\Phi \circ \Psi = \text{id}$ on \mathcal{C} .

Let ρ be a Υ -congruence on W and let $\Phi([K_\rho]) = \bar{\rho}$. We want to prove that $\rho = \bar{\rho}$.

Let $w_i, w_j \in W$ such that $w_i \rho w_j$, then for every $f \in K_\rho$, f is constant on the equivalence class $[w_i]_\rho$, i.e. $f(w_i) = f(w_j)$. Hence, $K_\rho(w_i, w_j) \cong G$ and $[w_i]_\rho \subseteq [w_i]_{\bar{\rho}}$. Vice versa, let $w_i \in W$ and suppose there exists $w_j \in W$ such that $w_j \notin [w_i]_\rho$, but $w_j \in [w_i]_{\bar{\rho}}$. Since $w_j \notin [w_i]_\rho$, there exists an $f \in K_\rho$ such that $f(w_i) = g$ and $f(w_j) = 1$, where $g \in G$ and $g \neq 1$. Then $K_\rho(w_i, w_j) = G \times G$ and this yields a contradiction.

We shall finally prove that $\Psi \circ \Phi = \text{id}$.

Let $K \in \mathcal{K}$, $\Phi([K]) = \rho_K$ and

$$\Psi(\Phi([K])) = [K_{\rho_K}].$$

Let $w_j \in [w_i]_{\rho_K}$. Since $K(w_i, w_j) \cong G$, it means that there exist automorphisms $\alpha_{w_i}, \alpha_{w_j} \in \text{Aut}(G)$ such that, for every $f \in K$, there exists $g \in G$ such that $f(w_i) = \alpha_{w_i}(g)$ and $f(w_j) = \alpha_{w_j}(g)$. We denote by $N_{\text{Sym}(\Delta)}(G)$ the normalizer of G in $\text{Sym}(\Delta)$. Since G acts regularly on Δ , for every $w \in W$ there exists n_w belonging to $N_{\text{Sym}(\Delta)}(G)$ such that $\alpha_w(g) = n_w^{-1} g n_w$, for $g \in G$. Consider the function $n : W \rightarrow N_{\text{Sym}(\Delta)}(G)$ given by $n(w) = n_w$. Let $F_{\rho_K} \in \mathcal{F}$ be a closed subgroup of $\text{Sym}(\Delta \times W)$ such that $K_{\rho_K} = F_{\rho_K} \cap G^W$. Since F_{ρ_K} is closed, $n^{-1} F_{\rho_K} n$ is closed. In fact, $n^{-1} F_{\rho_K} n \in \mathcal{F}$ and

$$K = n^{-1} K_{\rho_K} n = n^{-1} F_{\rho_K} n \cap G^W.$$

Since n is a bijection of $\Delta \times W$ which preserves the fibres of π , we have that $n^{-1} F_{\rho_K} n$ is isomorphic to F_{ρ_K} and then $[K] = [K_{\rho_K}]$. \blacksquare

Remark 12 *It is clear by the previous proof that in every class $[K] \in \mathcal{K}/R$ there exists $\bar{K} \in [K]$ which is constant on the equivalence classes of $\Phi([K])$.*

4 Special case

Let H be a group acting on a set X , $a \in X$ and $\Delta \subseteq X$. We denote by $a^H = \{ha : h \in H\}$, by $H_{(\Delta)}$ the pointwise stabilizer of Δ in H and by $H_{\{\Delta\}}$ the setwise stabilizer of Δ in H . We recall the following theorem, whose proof can be found in [4].

Theorem 13 ([4], Theorem 1.5A) *Let G be a group which acts transitively on a set Ω , and let $\alpha \in \Omega$. Let \mathcal{D} be the set of blocks Δ for G containing α , let \mathcal{H} denote the set of all subgroups H of G with $G_\alpha \leq H$. There is a bijection Ψ from \mathcal{D} onto \mathcal{H} given by $\Psi(\Delta) := G_{\{\Delta\}}$ whose inverse mapping Φ is given by $\Phi(H) := \alpha^H$. The mapping Ψ is order preserving in the sense that if $\Delta, \Theta \in \mathcal{D}$ then $\Delta \subseteq \Theta \iff \Psi(\Delta) \leq \Psi(\Theta)$.*

From now on let W be $\Omega^{(n)}$, the set of ordered n -tuples of distinct elements of the countable set Ω . Let $\Upsilon = \text{Sym}(\Omega)$ act on $\Omega^{(n)}$ in the natural way: let $\sigma \in \text{Sym}(\Omega)$, then $\sigma(a_1, \dots, a_n) = (\sigma(a_1), \dots, \sigma(a_n))$. In the sequel we denote $\text{Sym}(\Omega)$ by S when $\text{Sym}(\Omega)$ acts on Ω . Let ρ be a Υ -congruence, and $\Delta \subseteq \Omega^{(n)}$ be the equivalence class of ρ containing the element $\alpha = (a_1, \dots, a_n)$. We will refer to Δ as a block of imprimitivity containing α .

Definition 14 *Let $\alpha = (a_1, \dots, a_n) \in \Omega^{(n)}$. We define*

$$\text{supp}(\alpha) := \{a_1, \dots, a_n\}.$$

By Theorem 13, the subgroup $\Upsilon_{\{\Delta\}} = \{x \in \Upsilon \mid x\Delta = \Delta\}$ contains the stabilizer $\Upsilon_\alpha = S_{(a_1, \dots, a_n)}$. A proof of the following lemma can be found in [4].

Lemma 15 ([4] Lemma 8.4B) *Let Σ_1 and Σ_2 be subsets of an arbitrary set Ω such that $|\Sigma_1 \cap \Sigma_2| = |\Sigma_1| \leq |\Sigma_2|$. Then*

$$\langle \text{Sym}(\Sigma_1), \text{Sym}(\Sigma_2) \rangle = \text{Sym}(\Sigma_1 \cup \Sigma_2),$$

(we identify $\text{Sym}(\Sigma)$ with the pointwise stabilizer of $\Omega \setminus \Sigma$).

Proposition 16 *Let $\alpha = (a_1, \dots, a_n) \in \Omega^{(n)}$. Let $\Delta \neq \Omega^{(n)}$ be a block containing α . Let $\{\Gamma_i\}_{i \in I}$ be the set of finite subsets of Ω such that*

$$\Upsilon_\alpha \leq S_{(\Gamma_i)} \leq \Upsilon_{\{\Delta\}}.$$

Let $\Gamma = \bigcap_{i \in I} \Gamma_i$. Then

$$\Upsilon_\alpha \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}.$$

Moreover Γ is finite and $\Gamma \subseteq \{a_1, \dots, a_n\}$.

Proof. We notice that the index set I is non-empty: for instance the set $\{a_1, \dots, a_n\}$ belongs to $\{\Gamma_i\}_{i \in I}$. Moreover, it is finite since every $\Gamma_i \leq \{a_1, \dots, a_n\}$. In order to prove that $\Upsilon_\alpha \leq S_{(\Gamma)}$ it is sufficient to notice that for every $i \in I$, $\Gamma \subseteq \Gamma_i$. Then $\Upsilon_\alpha \leq S_{(\Gamma_i)} \leq S_{(\Gamma)}$, for every $i \in I$.

We use Lemma 15 to prove the inclusion $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$. Let $\Sigma_i = \Omega \setminus \Gamma_i$, for $i \in I$. Then by Lemma 15 we have $\langle S_{(\Gamma_i)}, i \in I \rangle = S_{(\bigcap_{i \in I} \Gamma_i)}$ and so $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$.

Notice that Γ is the smallest subset of Ω such that $\Upsilon_\alpha \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$. We want

to prove the set Γ has the smallest cardinality among the finite sets X of Ω such that $S_{(X)} \leq \Upsilon_{\{\Delta\}}$. Suppose not, then there exists a finite subset of Ω , say Σ , with $|\Sigma| \leq |\Gamma|$ and $S_{(\Sigma)} \leq \Upsilon_{\{\Delta\}}$.

If $\Gamma \cap \Sigma \neq \emptyset$, then by Lemma 15, we have

$$\Upsilon_\alpha \leq S_{(\Gamma)} \leq S_{(\Gamma \cap \Sigma)} \leq \Upsilon_{\{\Delta\}}$$

and, since Γ is the smallest subset of Ω such that $\Upsilon_\alpha \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$, this yields a contradiction.

If $\Gamma \cap \Sigma = \emptyset$, then

$$\langle S_{(\Gamma)}, S_{(\Sigma)} \rangle = S_{(\Gamma \cup \Sigma)} = S \leq \Upsilon_{\{\Delta\}}$$

but $\Upsilon_{\{\Delta\}} \neq S$, a contradiction. Thus, the set Γ has the smallest cardinality among the finite subsets X of Ω such that $S_{(X)} \leq \Upsilon_{\{\Delta\}}$.

Let $x \in \Upsilon_{\{\Delta\}}$, then we have $S_{(x\Gamma)} = x^{-1}S_{(\Gamma)}x \leq \Upsilon_{\{\Delta\}}$, and so, applying again Lemma 15 we get that $\Upsilon_{\{\Delta\}} \geq \langle S_{(\Gamma)}, S_{(x\Gamma)} \rangle = S_{(\Gamma \cap x\Gamma)}$. Thus, for all $x \in \Upsilon_{\{\Delta\}}$, $\Gamma = x\Gamma$ by the minimality of Γ and $\Upsilon_{\{\Delta\}} \leq S_{(\Gamma)}$.

To prove that $\Gamma \subseteq \{a_1, \dots, a_n\}$ it is sufficient to note that $S_{(\Gamma)} \geq \Upsilon_\alpha$, and the claim follows. \blacksquare

As the following result shows, a ρ -class in $\Omega^{(n)}$ can be a finite subset or an infinite subset of $\Omega^{(n)}$.

Proposition 17 *Let $\Delta \neq \Omega^{(n)}$ be the equivalence class of a Υ -congruence ρ containing the element $(a_1, \dots, a_n) \in \Omega^{(n)}$. Then*

- a) Δ is finite if and only if $S_{(a_1, \dots, a_n)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{a_1, \dots, a_n\}}$;
- b) Δ is a countably infinite set if and only if $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}$, for some finite set $\Gamma \subsetneq \{a_1, \dots, a_n\}$.

Proof.

a) Suppose Δ is a finite set in $\Omega^{(n)}$. If it doesn't exist any $\Gamma \subsetneq \{a_1, \dots, a_n\}$ such that $S_{(a_1, \dots, a_n)} < S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$, by Proposition 16, since $S_{(a_1, \dots, a_n)} \leq \Upsilon_{\{\Delta\}}$, we have $S_{(a_1, \dots, a_n)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{a_1, \dots, a_n\}}$.

Hence, suppose that there exists a finite set $\Gamma \subsetneq \{a_1, \dots, a_n\}$ such that $S_{(a_1, \dots, a_n)} \leq S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$. Let $x \in S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$, then $x\Delta = \Delta$. Take $a_i \in \{a_1, \dots, a_n\} \setminus \Gamma$. Then pick $a \in \Omega$ such that $a \notin \text{supp}(\delta)$, for every $\delta \in \Delta$. By k -transitivity of S , for any $k \in \mathbb{N}$, it is possible to choose an element x in $S_{(\Gamma)}$, such that $x(a_i) = a$. Then

$$x(a_1, \dots, a_i, \dots, a_n) = (x(a_1), \dots, a, \dots, x(a_n)) \in \Delta.$$

But this yields a contradiction, since $a \notin \text{supp}(\delta)$, for every $\delta \in \Delta$.

In the other direction, if $S_{(a_1, \dots, a_n)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{a_1, \dots, a_n\}}$ then $\Delta = (a_1, \dots, a_n)^{\Upsilon_{\{\Delta\}}} \subseteq (a_1, \dots, a_n)^{S_{\{a_1, \dots, a_n\}}}$, and $|(a_1, \dots, a_n)^{S_{\{a_1, \dots, a_n\}}}|$ is finite.

b) We now assume Δ is a countably infinite set. Suppose there does not exist any finite set $\Gamma \subsetneq \{a_1, \dots, a_n\}$ such that $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}}$. By Theorem 13 we have that $S_{(a_1, \dots, a_n)} \leq \Upsilon_{\{\Delta\}}$. Since for every finite set $\Gamma \subsetneq \{a_1, \dots, a_n\}$ we have $S_{(\Gamma)} \not\leq \Upsilon_{\{\Delta\}}$, then $\{a_1, \dots, a_n\}$ is the smallest subset of Ω such that $S_{(a_1, \dots, a_n)} \leq \Upsilon_{\{\Delta\}}$ and so, by Proposition 16, $\Upsilon_{\{\Delta\}} \leq S_{\{a_1, \dots, a_n\}}$. Take an element (b_1, \dots, b_n) of Δ , such that $\{b_1, \dots, b_n\} \neq \{a_1, \dots, a_n\}$; as Δ is infinite, this element there exists. By the n -transitivity of S , there exists an element $x \in S$ such that $x(a_1) = b_1, \dots, x(a_n) = b_n$. Then $x(a_1, \dots, a_n) \in \Delta$ and so we have an element $x \in \Upsilon_{\{\Delta\}}$ but not in $S_{\{a_1, \dots, a_n\}}$. This yields a contradiction.

Conversely suppose $\Gamma \subsetneq \{a_1, \dots, a_n\}$, and $S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}}$. Then $(a_1, \dots, a_n)^{S_{(\Gamma)}} \subseteq \Delta$, and since $(a_1, \dots, a_n)^{S_{(\Gamma)}}$ is infinite, then Δ is infinite. ■

Remark 18 If $|\Gamma| = n$, $n \geq 1$, then $S_{(\Gamma)} \trianglelefteq S_{\{\Gamma\}}$ and $S_{\{\Gamma\}}/S_{(\Gamma)} \cong \text{Sym}_n$ the symmetric group on n points. Given an element $\alpha = (a_1, \dots, a_n) \in \Omega^{(n)}$ and a finite block Δ containing it, we have that $H = \Upsilon_{\{\Delta\}}$ satisfies the following inclusions: $S_{(\Gamma)} \leq H \leq S_{\{\Gamma\}} \leq S$, where $\Gamma = \{a_1, \dots, a_n\}$. Then $H/S_{(\Gamma)}$ is isomorphic to a subgroup of Sym_n . There exists a bijection Θ between the subgroups of Sym_n and the subgroups of $S_{\{\Gamma\}}$ which contain $S_{(\Gamma)}$.

We shall denote by

$$\mathcal{K}_F = \{K \in \mathcal{K} \mid \rho_K \text{ has finite equivalence classes}\}.$$

Proposition 19 Let \mathcal{L} be the set of subgroups of Sym_n . Then there exists a bijection

$$\zeta : \mathcal{K}_F/R \rightarrow \mathcal{L}.$$

Proof. By Theorem 11, it is sufficient to find a bijection between the set of finite blocks containing an element $\alpha = (a_1, \dots, a_n)$ and \mathcal{L} . Let Δ be a finite block in $\Omega^{(n)}$ containing α . We have that

$$\Upsilon_\alpha = S_{(\Gamma)} \leq \Upsilon_{\{\Delta\}} \leq S_{\{\Gamma\}} \leq S$$

where $\Gamma = \text{supp}(\alpha)$. Then by Remark 18, $\Upsilon_{\{\Delta\}}$ is the image by Θ of a subgroup of Sym_n . If $\Delta_1 \neq \Delta_2$ then $\Upsilon_{\{\Delta_1\}} \neq \Upsilon_{\{\Delta_2\}}$. By Remark 18, it follows that the map ζ is injective. In the other direction, let $H \in \mathcal{L}$. By the remark 18, $\Theta(H)$ is a subgroup L of $S_{\{\Gamma\}}$ which contains $\Upsilon_\alpha = S_{(\Gamma)}$. Then, by Theorem 13, we have a finite block α^L containing α . ■

Proposition 20 Let $\alpha = (a_1, \dots, a_n) \in \Omega^{(n)}$ and let \mathcal{D}_F^α be the set of the finite blocks in $\Omega^{(n)}$ containing α . Then the elements of \mathcal{D}_F^α are exactly the sets α^H , where H is a subgroup of $\text{Sym}\{a_1, \dots, a_n\}$.

Proof. Let $\Delta \in \mathcal{D}_F^\alpha$. Let H' be the subgroup of $\text{Sym}(\Omega)$ such that $\alpha^{H'} = \Delta$. Then

$$\Upsilon_\alpha \leq H' \leq S_{\{\Gamma\}},$$

where $\Gamma = \text{supp}(\alpha)$. Since $S_{\{\Gamma\}}/S_{(\Gamma)} \cong \text{Sym}_n$ we have that $H' = H \times \text{Sym}(\Omega \setminus \Gamma)$, where H is a subgroup of $\text{Sym}\{a_1, \dots, a_n\}$. Then $\Delta = \alpha^H$. Viceversa, taken a subgroup $H \leq \text{Sym}\{a_1, \dots, a_n\}$, $\alpha^H = \alpha^{H \times \text{Sym}(\Omega \setminus \Gamma)}$. By Theorem 13 α^H is a block in $\Omega^{(n)}$. ■

The same argument works for the following:

Proposition 21 Let $\alpha = (a_1, \dots, a_n) \in \Omega^{(n)}$ and let \mathcal{D}_I^α be the set of non-trivial infinite blocks in $\Omega^{(n)}$ containing α . Then the elements of \mathcal{D}_I^α are exactly the sets $\alpha^{L \times \text{Sym}(\Omega \setminus \Xi)}$, where $\Xi \subsetneq \{a_1, \dots, a_n\}$ and L is a subgroup of $\text{Sym}(\Xi)$.

Let us mention a little remark about Proposition 20. Let $\alpha = (a_1, \dots, a_n)$. Denote $\text{Sym}\{a_1, \dots, a_n\}$ by Sym_n . Consider the set

$$\alpha^{\text{Sym}_n} = \{\sigma(a_1, \dots, a_n), \sigma \in \text{Sym}_n\}.$$

Let $[K] \in \mathcal{K}_F/R$, and $\bar{K} \in [K]$ be the subgroup of G^W such that is constant on the equivalence classes of $\Phi(K)$ (remind Remark 12). By Proposition 20 there exists a subgroup T of Sym_n such that K restricted to $\Delta = \alpha^T$ is constant on it. The system of blocks containing Δ is the set $\{g\Delta, g \in \text{Sym}(\Omega)\}$. We look at the restriction of \bar{K} to the set α^{Sym_n} . This is the subgroup of $G^{\alpha^{\text{Sym}_n}}$ of the function from α^{Sym_n} to G constant on the subsets $bT(\alpha)$, where bT are the left cosets of T in Sym_n . We notice that the cardinalities of the finite blocks in $\Omega^{(n)}$ are exactly the cardinalities of the subgroups of Sym_n .

5 Commentary

5.1 Finite Covers

As is well known, a subgroup of $\text{Sym}(W)$ is closed if and only if it is the group of automorphisms of some first-order structure with domain W (see for instance Proposition (2.6) in [3]). Thus we state the following definition.

A *permutation structure* is a pair $\langle W, G \rangle$, where W is a non-empty set (the *domain*), and G is a closed subgroup of $\text{Sym}(W)$. We refer to G as the automorphism group of W . If A and B are subsets of W (or more generally of some set on which $\text{Aut}(W)$ acts), we shall refer to $\text{Aut}(A/B)$ as the group of permutations of A which extend to elements of $\text{Aut}(W)$ fixing every element of B and to $\text{Aut}(A/\{B\})$ as the group of permutations of A which extend to elements of $\text{Aut}(W)$ stabilizing setwise the set B .

Permutation structures are obtained by taking automorphism groups of first-order structures and we often regard a first-order structure as a permutation structure without explicitly saying so. Let $\pi : C \rightarrow W$ be a finite cover (Definition 1), we frequently use the notation $C(w)$ to denote the fibre $\pi^{-1}(w)$ above w in the cover $\pi : C \rightarrow W$.

We recall that the *fibre group* $F(w)$ of π on $C(w)$ is $\text{Aut}(C(w)/w)$, while the *binding group* $B(w)$ of π on $C(w)$ is $\text{Aut}(C(w)/W)$. It follows that the binding group is a normal subgroup of the fibre group. If $\text{Aut}(W)$ acts transitively on W , then all the fibre groups are isomorphic as permutation groups, as are the binding groups. There is a continuous epimorphism $\chi_w : \text{Aut}(W/w) \rightarrow F(w)/B(w)$ called *canonical epimorphism* (Lemma 2.1.1 [6]). Thus if $\text{Aut}(W/w)$ has no proper open subgroup of finite index, then $F(w) = B(w)$.

Let $\pi_1 : C_1 \rightarrow W$ and $\pi_2 : C_2 \rightarrow W$ be two finite covers of W . Then π_1 is said to be *isomorphic* to π_2 if there exists a bijection $\alpha : C_1 \rightarrow C_2$ with $\alpha(\pi_1^{-1}(w)) = \pi_2^{-1}(w)$ for all $w \in W$, such that the induced map $f_\alpha : \text{Sym}(C_1) \rightarrow \text{Sym}(C_2)$ satisfies $f_\alpha(\text{Aut}(C_1)) = \text{Aut}(C_2)$.

The *Cover Problem* is, given W and data $(F(w), B(w), \chi_w)$, to determine (up to isomorphism) the possible finite covers with these data.

If C and C' are permutation structures with the same domain and $\pi : C \rightarrow W$, $\pi' : C' \rightarrow W$ are finite covers with $\pi(c) = \pi'(c)$ for all $c \in C = C'$, we say that π' is a *covering expansion* of π if $\text{Aut}(C') \leq \text{Aut}(C)$.

Suppose that C and W are two permutation structures and $\pi : C \rightarrow W$ is a

finite cover. The cover is *free* if

$$\text{Aut}(C/W) = \prod_{w \in W} \text{Aut}(C(w)/W),$$

that is, the kernel is the full direct product of the binding groups.

The existence of a free finite cover with prescribed data depends on the existence of a certain continuous epimorphism.

Indeed, let W be a transitive permutation structure and $w_0 \in W$. Given a permutation group F on a finite set X , a normal subgroup B of F and a continuous epimorphism

$$\chi : \text{Aut}(W/w_0) \rightarrow F/B,$$

then there exists a free finite cover $\sigma : M \rightarrow W$ with fibre and binding groups at w_0 equal to F and B , and such that the canonical epimorphism χ_{w_0} is equal to χ . With these properties σ is determined uniquely (see [6], Lemma 2.1.2).

A *principal* cover $\pi : C \rightarrow W$ is a free finite cover where the fibre and binding groups at each point are equal. Free covers are useful in describing finite covers with given data because every finite cover $\pi : C \rightarrow W$ is an expansion of a free finite cover with the same fibre groups, binding groups and canonical homomorphisms as in π (see [6], Lemma 2.1.3).

Let's go back to Section 2. Using the language of finite covers, \mathcal{F} is the set of the expansions of the principal finite covers of $\langle W, \Upsilon \rangle$, with all fibre groups and binding groups equal to a given group G .

In the case when G is a simple non-abelian regular group, our main theorem shows that the Υ -congruences on W describe (up to isomorphisms over W) the kernels of expansions belonging to \mathcal{F} .

5.2 Bi-interpretability

Definition 22 *Two permutation structures are bi-interpretable if their automorphism groups are isomorphic as topological groups.*

For a model-theoretic interpretation, if the permutation structures arise from \aleph_0 -categorical structures, see Ahlbrandt and Ziegler ([1]). Usually classification of structures is up to bi-interpretability.

Let $n \in \mathbb{N}$. Consider $\Omega^{(n)}$ as a first-order structure with automorphism group equal to $\text{Sym}(\Omega)$.

Proposition 23 *Let $M_1 := \Delta \times \Omega^{(n)}$ and $\pi_1 : M_1 \rightarrow \Omega^{(n)}$ be an expansion of a principal finite cover of $\Omega^{(n)}$ with all binding groups equal to a simple non-abelian finite group G acting on Δ . Let K_1 be the kernel of π_1 .*

Suppose that the congruence classes which K_1 determine have finite cardinality. Then, $\forall m > n$ there exists a permutation structure $M_2 := \Delta \times \Omega^{(m)}$ and a finite cover $\pi_2 : M_2 \rightarrow \Omega^{(m)}$ with all fibre groups and binding groups equal to G such that M_1 is bi-interpretable with M_2 and the kernel K_2 of π_2 determines a $\text{Sym}(\Omega)$ -congruence with equivalence classes of infinite cardinality.

Proof. By the notation $M_1(\alpha)$, we mean the copy of Δ over the element $\alpha \in \Omega^{(n)}$. The kernel K_1 , by Lemma 5, determines a $\text{Sym}(\Omega)$ -congruence ρ which, by hypothesis has equivalence classes of finite cardinality. Let m be a positive integer greater then n and M_2 be the set

$$M_2 = \{(w, m) : w = (\alpha, c_1, \dots, c_{m-n}) \text{ and } m \in M_1(\alpha)\}$$

where $\alpha \in \Omega^{(n)}$ and $c_1, \dots, c_{m-n} \in \Omega \setminus \text{supp}(\alpha)$ and are all distinct. Obviously $M_2 = \Delta \times \Omega^{(m)}$. Let $\mu_1 : \text{Aut}(M_1) \rightarrow \text{Sym}(\Omega)$ be the map induced by π_1 and Λ be the subgroup of $\text{Sym}(\Omega) \times \text{Aut}(M_1)$

$$\Lambda = \{(g, \sigma) : g = \mu_1(\sigma)\}.$$

Our claim is to show that $\langle M_2, \Lambda \rangle$ is a permutation structure and that $\pi_2 : M_2 \rightarrow \Omega^{(m)}$ given by $\pi_2(w, m) = w$ is a finite cover of $\Omega^{(m)}$ with $F(w) = B(w) = G$ and kernel K_2 which determines a $\text{Sym}(\Omega)$ -congruence with equivalence classes of infinite cardinality .

It is easy to check that Λ is a permutation group on M_2 which preserves the partition of M_2 given by the fibres of π_2 .

We equip $\text{Sym}(\Omega) \times \text{Aut}(M_1)$ with the product topology. This topology coincides with the topology of the pointwise convergence induced by $\text{Sym}(\Omega^{(m)} \times M_1)$ on $\text{Sym}(\Omega) \times \text{Aut}(M_1)$. The map Φ given by

$$\text{Sym}(\Omega) \times \text{Aut}(M_1) \xrightarrow{p_1} \text{Sym}(\Omega)$$

and the map Ψ given by

$$\text{Sym}(\Omega) \times \text{Aut}(M_1) \xrightarrow{p_2} \text{Aut}(M_1) \xrightarrow{\mu_1} \text{Sym}(\Omega)$$

where p_1 and p_2 are the projections on the first and second component, respectively, are continuous. The permutation group Λ is equal to the difference kernel

$$Z = \{(g, \sigma) \in \text{Sym}(\Omega) \times \text{Aut}(M_1) : \Psi(g, \sigma) = \Phi(g, \sigma)\}$$

which, by Proposition 3 pag. 30 of [8], is closed in $\text{Sym}(\Omega) \times \text{Aut}(M_1)$. Moreover, $\text{Sym}(\Omega) \times \text{Aut}(M_1)$ is closed in $\text{Sym}(\Omega^{(m)} \times M_1)$ and then $\langle M_2, \Lambda \rangle$ is a permutation structure. The usual map induced by π_2

$$\mu_2 : \Lambda \rightarrow \text{Sym}(\Omega^{(m)})$$

has image $\text{Sym}(\Omega)$. The kernel of μ_2 , which we denote by K_2 , is

$$K_2 = \{(id, \sigma) \in \Lambda : \sigma \in K_1\}.$$

Then $K_1 \cong K_2$. Let $(w, m) = (\alpha, c_1, \dots, c_{m-n}, m) \in M_2$ where $\alpha \in \Omega^{(n)}$ and $c_1, \dots, c_{m-n} \in \Omega \setminus \text{supp}(\alpha)$ and are all distinct. Let (id, σ) be an element in K_2 . If we restrict it to the fibre over w , we see that it is the same as restricting σ to the fibre over α . Hence the binding group over w , $B_2(w)$, is clearly isomorphic to G . The same holds for the fibre group: let $w = (\alpha, c_1, \dots, c_{m-n})$, then $F_2(w)$ over is the restriction of the group

$$\text{Aut}(M_2/w) = \{(g, \sigma) \in \Lambda : g \in \text{Sym}(\Omega)_{((\alpha, c_1, \dots, c_{m-n}))}\}$$

to the fibre over w . Since $g \in \text{Sym}(\Omega)_{((\alpha, c_1, \dots, c_{m-n}))}$ then $g \in \text{Sym}(\Omega)_{(\alpha)}$. Hence $\sigma \in \text{Aut}(M_1/\alpha)$ and so $F_2(w)$ is isomorphic to G .

Moreover, if we consider two points of $\Omega^{(m)}$, say $w = (\alpha, c_1, \dots, c_{m-n})$ and $w' = (\alpha', c'_1, \dots, c'_{m-n})$, with $\alpha\rho\alpha'$, we have that $K_2(w, w') \cong G$. Vice versa if $K_2(w, w') \cong G$, it means that $K_1(\alpha, \alpha') \cong G$. Then the $\text{Sym}(\Omega)$ -congruence, ρ' , that K_2 determines is given by $w\rho'w'$ if and only if $\alpha\rho\alpha'$. In the equivalence class of $w = (\alpha, c_1, \dots, c_{m-n})$ for instance there are all the elements of the form $(\alpha, c'_1, \dots, c'_{m-n})$, with $c_1, \dots, c_{m-n} \in \Omega \setminus \text{supp}(\alpha)$ and pairwise distinct. Then the equivalence classes of ρ' are of infinite cardinality.

Next we check the bi-interpretability. We consider the map

$$\begin{aligned} \beta : \quad \Lambda &\rightarrow \text{Aut}(M_1) \\ (g, \sigma) &\mapsto \sigma \end{aligned}$$

The kernel of β is $\ker\beta = \{(g, id) \in \Lambda : g = \mu_1(id)\}$. Then β is injective. It is also surjective since, given $\sigma \in \text{Aut}(M_1)$, $(\mu_1(\sigma), \sigma) \in \Lambda$. Clearly the inverse map is given by $\beta^{-1}(\sigma) = (\mu_1(\sigma), \sigma)$.

It is a topological isomorphism. Indeed, take a basic open neighbourhood of the identity in $\text{Aut}(M_1)$, say $\text{Aut}(M_1)_{(\Gamma)}$, where $\Gamma = \{m_i\}_{i \in I}$ is a finite set of M_1 . Each $m_i \in M_1(\alpha_i)$. Then

$$\beta^{-1}(\text{Aut}(M_1)_{(\Gamma)}) = \{(\mu_1(\sigma), \sigma) : \sigma \in \text{Aut}(M_1)_{(\Gamma)}\}.$$

For each α_i , we choose $c_1^i, \dots, c_{m-n}^i \in \Omega$ such that $w_i = (\alpha_i, c_1^i, \dots, c_{m-n}^i)$ is a fulfillment of α_i to an element of $\Omega^{(m)}$. The map

$$\begin{aligned} \beta^{-1} : \quad \text{Aut}(M_1) &\rightarrow \text{Sym}(\Omega) \times \text{Aut}(M_1) \\ \sigma &\mapsto (\mu_1(\sigma), \sigma) \end{aligned}$$

is continuous. The image of β^{-1} is Λ and Λ has the topology induced by $\text{Sym}(\Omega) \times \text{Aut}(M_1)$, then $\beta^{-1} : \text{Aut}(M_1) \rightarrow \Lambda$ is continuous. Hence, we have proved the bi-interpretability. \blacksquare

5.3 Almost-free finite covers

Let W be a transitive structure and ρ be an $\text{Aut}(W)$ -congruence on W . Given a ρ -equivalence class $[w]$, we denote by $C([w]) = \bigcup_{w_i \in [w]} C(w_i)$, by $F([w])$ the permutation group induced by $\text{Aut}(C/\{[w]\})$ on $C([w])$, and by $B([w])$ the permutation group induced by the kernel of π on $C([w])$.

Note that $B([w]) \leq F([w])$.

Lemma 24 *Suppose that W is a transitive structure and ρ an $\text{Aut}(W)$ -congruence on W . Let $\pi : C \rightarrow W$ be a finite cover. Then, for every ρ -class $[w]$ in W*

1. *there exists a finite-to-one surjection*

$$\pi_{[w]} : C([w]) \rightarrow [w]$$

such that its fibres form an $F([w])$ -invariant partition of $C([w])$;

2. *there is a continuous epimorphism*

$$\chi_{[w]} : \text{Aut}(W/\{[w]\}) \rightarrow F([w])/B([w]).$$

Proof. The first point is clear.

The second point require a little proof. Let $g \in \text{Aut}(W/\{[w]\})$. Then there exists $h \in \text{Aut}(C/\{[w]\})$ which extends g . Let $\psi : \text{Aut}(W/\{[w]\}) \rightarrow \text{Aut}(C/\{[w]\})/\text{Aut}(C/W)$ be the map defined by $\psi(g) = h \text{Aut}(C/W)$. This map is well defined. Suppose that also \bar{h} extends g . Then $h^{-1}\bar{h} \in \text{Aut}(C/W)$ and so $h \text{Aut}(C/W) = \bar{h} \text{Aut}(C/W)$. Consider the restriction to the set of fibres over $\{[w]\}$. So we have a map $\xi_{[w]} : \text{Aut}(C/\{[w]\})/\text{Aut}(C/W) \rightarrow \text{Sym}(C([w])/B([w]))$, given by $\xi_{[w]}(h \text{Aut}(C/W)) = h|_{C([w])}B([w])$, which is clearly onto on $F([w])/B([w])$. Let $g \in \text{Aut}(W/\{[w]\})$. We define $\chi_{[w]}(g) := \xi_{[w]}\psi(g)$. In order to prove that $\chi_{[w]}$ is continuous, we show that ψ and $\xi_{[w]}$ are continuous.

The restriction map $\xi_{[w]}$ is continuous by Lemma 1.4.1 of [6]. Consider $\text{Sym}(C([w])/B([w]))$ with the topology of pointwise convergence and $\text{Sym}(C([w])/B([w]))$ with the quotient topology. Let $\mu|_{\text{Aut}(C/\{[w]\})} : \text{Aut}(C/\{[w]\}) \rightarrow \text{Aut}(W/\{[w]\})$ be the map induced by μ . Since $[w]$ is a ρ -equivalence class $\text{Aut}(C/\{[w]\})$ is an open subgroup of $\text{Aut}(C)$. Indeed, let $c \in C([w])$. Take $h \in \text{Aut}(C/c)$. Then $h(C([w])) = C([w])$. If $g = \mu(h)$, we have $g(w) = w$, and being $[w]$ a $\text{Aut}(W)$ -congruence class, this implies that $g([w]) = [w]$. Hence $h \in \text{Aut}(C/\{[w]\})$. By Lemma 7 we have that $\text{Aut}(C/\{[w]\})$ is an open subgroup of $\text{Aut}(C)$. By the same reasoning we get that $\text{Aut}(W/\{[w]\})$ is open in $\text{Aut}(W)$. Now, since μ is open also $\mu|_{\text{Aut}(C/\{[w]\})}$ will be open. Hence by Proposition 1, pag 21 of [8], we have the continuity of ψ . ■

Definition 25 Let $\pi : C \rightarrow W$ be a finite cover of W , $w \in W$, with binding groups isomorphic to a group G and kernel K . We shall say that π is **almost free** with respect to ρ if

1. $K([w]) \cong G$ for each $[w] \in W/\rho$
2. $K(w_1, w_2) \cong G \times G$ for each $w_2 \notin [w_1]$.

A class of almost free finite cover is the set of the expansions of the free finite covers with binding groups isomorphic to a simple non-abelian group G .

Let $R := W/\rho$. Given a transitive structure W and an $\text{Aut}(W)$ -congruence ρ , naturally we have an induced map

$$M : \text{Aut}(W) \rightarrow \text{Sym}(R).$$

The map M is continuous, but the image of $\text{Aut}(W)$ by M is not necessarily closed. The following counterexample is due to Peter Cameron (private communication).

Take the generic bipartite graph B , and consider the group G of automorphisms fixing the two bipartite blocks, acting on the set of edges of the graph. On the set of edges there are two equivalence relations, "same vertex in the first bipartite block", and "same vertex in the second bipartite block". Clearly G is precisely the group preserving these two equivalence relations, and so is closed. But the group induced on the set of equivalence classes of each relation is highly transitive and not the symmetric group, therefore not closed.

Proposition 26 *Let W be a transitive structure and ρ an $\text{Aut}(W)$ -congruence on W . We suppose that the following assumptions hold:*

1. *Let F be a closed permutation group on a set X . Fix $w_0 \in W$ and let $[w_0]$ be the ρ -equivalence class of w_0 . Suppose that there exists a finite -to-one surjection*

$$\sigma : X \rightarrow [w_0]$$

such that the fibres form an F -invariant partition of X and that the induced map $T : F \rightarrow \text{Sym}([w_0])$ has image $\text{Aut}(W/\{[w_0]\})_{\{[w_0]\}}$. Let B the kernel of T .

2. *The map T induces a map*

$$\chi : \text{Aut}(W/\{[w_0]\}) \rightarrow F/B$$

defined as $\chi(g) = hB$, where $h \in F$ and $T(h) = g|_{[w]}$. Assume that χ is continuous.

3. *Let G be the permutation group induced by B on $\sigma^{-1}(w_0)$. Suppose that B is isomorphic to G .*
4. *Assume that the map M is injective, open and with closed image.*

Then there exists an almost free finite cover π_0 of W with respect to ρ with binding groups isomorphic to G , $F([w_0]) = F$, $B([w_0]) = B$ and map $\chi_{[w_0]}$ equal to χ . Moreover, if $\tilde{\pi}_0$ is an almost free finite cover with respect to ρ with $F([w_0])$ and $B([w_0])$ isomorphic as permutation groups to F and B respectively, and $\chi_{[w_0]}$ equal to χ (up to isomorphism), then $\tilde{\pi}_0$ is isomorphic over W to π_0 .

Proof. This is an application of Lemma 2.1.2 in [6]. In this proof we will deal with a map with all the properties of a finite cover but the finiteness condition on the fibres (hence we allow the cover to having fibres of infinite cardinality). We shall call such a map a cover.

We give to R the first-order structure with automorphism group the image of M . Let $r_0 = [w_0]$. We have that

$$M^{-1} : \text{Aut}(R/r_0) \rightarrow \text{Aut}(W/\{[w_0]\})$$

is continuous. Then we have a continuous map $\chi : \text{Aut}(R/r_0) \rightarrow F/B$.

Since we are going to use a slightly changed version of the proof of Lemma 2.1.2 in [6] and then to use specific steps out of it, we are going to give the general lines of the proof for the use of the reader. For the details we refer to the book [6].

We are going to sketch the proof of the following statement: Let R be a transitive permutation structure and $r_0 \in R$. Let F be a closed permutation group on a set X , and B be a normal subgroup of F . Suppose there is a continuous epimorphism $\chi : \text{Aut}(R/r_0) \rightarrow F/B$. Then there exists a cover $\pi : M \rightarrow R$ with fibre group and binding group at r_0 respectively equal to F and B and canonical epimorphism at r_0 equal to χ . Moreover, if ν is a free cover with with $F(r_0)$ and $B(r_0)$ isomorphic as permutation groups to F and B respectively, and χ_{r_0} equal to χ (up to isomorphism), then ν is isomorphic over R to π .

The proof is made essentially in three steps. First the following cover is constructed.

Let C be the set of left cosets of $\ker \chi$ in $\text{Aut}(R)$. Consider the map $\theta : C \rightarrow R$ given by $\theta(g \ker \chi) = gr_0$. The permutation group $\text{Aut}(R)$ induces a group of permutation on C . The induced group is a closed subgroup of $\text{Sym}(C)$ and so we can consider C as a relational structure with automorphism group isomorphic to $\text{Aut}(R)$. Then the map θ is a cover with trivial kernel.

Let $Y = \theta^{-1}(r_0) \cup X$. Put on Y the relational structure which has as automorphism group F : the action of $h \in F$ on $m \in \theta^{-1}(r_0)$ is $h(m) = (\chi^{-1}(hG))(m)$. For every $r \in R$ choose $g_r \in \text{Aut}(R)$ such that $g_r r = r_0$ (with $g_{r_0} = \text{id}$). Then $g_r(\theta^{-1}(r)) = \theta^{-1}(r_0)$ and it induces an embedding $\eta_r : \theta^{-1}(r) \rightarrow Y$.

The second step is the following: we built a cover $\pi' : M' \rightarrow R$, where the domain of M' is made of the disjoint union of R , C and $R \times Y$ and π' is the identity on R , θ on C , the projection to the first coordinate on $R \times Y$. We also have an injection $\tau : C \rightarrow R \times Y$ given by $\tau(c) = (r, \eta_r(c))$, whenever $\theta(c) = r$. Moreover, the structure of M' is made up of the original structure on R and C and for each n -ary relation R on Y we have an n -ary relation R' on $R \times Y$ given by

$$R'((r_1, y_1), \dots, (r_n, y_n)) \text{ iff } r_i = r_j, \forall i, j \text{ and } R(y_1, \dots, y_n).$$

Now we see how to extend an automorphism of R to a permutation of M' which preserves the above structure.

Let $g \in \text{Aut}(R)$, then we get an automorphism of C . Let $gr_1 = r_2$, then via τ we have a bijection from $\{r_1\} \times \theta^{-1}(r_0)$ to $\{r_2\} \times \theta^{-1}(r_0)$. In fact, let $\bar{g} \ker \chi \in \theta^{-1}(r_0)$, then $\tau g \tau^{-1}(\bar{g} \ker \chi) = g_{r_2} g g_{r_1}^{-1} \bar{g} \ker \chi$. Since $g_{r_2} g g_{r_1}^{-1} \in \text{Aut}(W/\{r_0\})$, if we choose a representative h in the class $\chi(g_{r_2} g g_{r_1}^{-1})$ then $h(\bar{g} \ker \chi) = \tau g \tau^{-1}(\bar{g} \ker \chi)$ and this extends to a permutation $\beta(r, g)$ of Y . If we also denote by $\beta(r, g)$ the induced map from $r \times Y$ to $gr \times Y$, then $\omega(g) = g \cup \bigcup_{r \in R} \beta(r, g)$ is a permutation of M' which preserves the structure we put above on M' and extends g .

Let π the restriction of π' to $M = R \times X$ considered as permutation structure with $\text{Aut}(M')$ acting. Then $\pi : M \rightarrow R$ is a free cover of R and kernel isomorphic to G^R .

Now the uniqueness, the third step. Let $\gamma : N(w_0) \rightarrow X$ be the bijection which gives rise to the isomorphism (we call it $\tilde{\gamma}$) as permutation groups between $F(B)$ and $F(w_0)$ ($B(w_0)$). Let $\nu : N \rightarrow R$ be a cover with $F(w_0)$ and $B(w_0)$ isomorphic as permutation groups to F and B respectively and $\chi_{w_0} = \tilde{\gamma} \circ \chi$. For each $r \in R$, g_r can be extended to an automorphism $\hat{g}_r \in \text{Aut}(N)$. We define the map $\beta : N \rightarrow R \times X$ in the following way: if $n \in \nu^{-1}(r)$, define $\beta(n) := (r, \gamma(\hat{g}_r(n))) \in R \times X$. As it is shown in Lemma 2.1.2 in [6], this is a bijection which gives rise to an isomorphism of covers.

Let $g_r \in \text{Aut}(R)$ be the permutations used above for constructing the free cover M . Then we construct a finite cover of W in the following way. Consider the set

$$C_0 := \{(w, k) : w \in r \text{ and } k \in \sigma^{-1}(M^{-1}(g_r)(w))\}$$

The map $\pi_0 : C_0 \rightarrow W$ given by $\pi_0(w, k) = w$ is a finite-to-one surjection. Let $\alpha : R \times X \rightarrow C_0$ be the map defined in the following way: let $k \in X$, then

there exists $w \in [w_0]$ such that $k \in \sigma^{-1}(w)$. We define

$$\alpha(r, k) := (M^{-1}(g_r^{-1})w, k).$$

This is a bijection. Let $f_\alpha : \text{Sym}(M) \rightarrow \text{Sym}(C_0)$ be the induced map by α . The image by f_α of $\text{Aut}(M)$ is closed in $\text{Sym}(C_0)$. We denote it by $\text{Aut}(C_0)$. Let $C_0(w)$ be the fibre over w of π_0 . If $w \in r_1$ then $C_0(w) = \sigma^{-1}(M^{-1}(g_{r_1})(w))$. We have that $\alpha^{-1}C_0(w) = (r_1, \sigma^{-1}(M^{-1}(g_{r_1})(w)))$.

Take an element g of $\text{Aut}(M)$. We are going to show that $\alpha g \alpha^{-1}$ preserves the partition of C_0 given by the fibres of π_0 .

Let $\bar{g} \in \text{Aut}(W)$ such that $M(\bar{g})$ is the induced permutation on R by g . If $M(\bar{g})r_1 = r_2$, there exists $f \in F$ such that

$$g(r_1, \sigma^{-1}(M^{-1}(g_{r_1})w)) = f(\sigma^{-1}(M^{-1}(g_{r_1})w) = \sigma^{-1}(M^{-1}(g_{r_2})\bar{g}w).$$

By the proof of Lemma 2.1.2 in [6], we see that the element f is a representative of the class $\chi(M^{-1}(g_{r_2})\bar{g}M^{-1}(g_{r_1}^{-1}))$. Hence $g(r_1, \sigma^{-1}(M^{-1}(g_{r_1})w)) = (r_2, \sigma^{-1}(M^{-1}(g_{r_2})\bar{g}w))$ and then

$$\alpha g \alpha^{-1}C_0(w) = C_0(\bar{g}w),$$

i.e. the fibres of π_0 form an $\text{Aut}(C_0)$ -invariant partition of C_0 .

Let $\mu_0 : \text{Aut}(C_0) \rightarrow \text{Sym}(W)$ be the induced homomorphism. Take an element $g \in \text{Aut}(W)$ and an extension $\tilde{g} \in \text{Aut}(M)$ of $M(g)$. The argument above shows as well that the $\text{Im}\mu_0$ is equal to $\text{Aut}(W)$. The kernel of μ_0 is $\alpha \ker \pi \alpha^{-1}$. It is isomorphic to G^R . Since $\ker \pi$ induced on $\sigma^{-1}(w)$ and on X is isomorphic to G , then $\ker \pi_0$ induced on any fibre of π_0 and on $C_0([w_0])$ is isomorphic to G as well. So we have an almost free finite cover $\pi_0 : C_0 \rightarrow W$ as required.

Let $\nu_0 : N_0 \rightarrow W$ be a finite cover of W with binding groups isomorphic to a finite group G with kernel isomorphic to G^R and with $B([w_0]) \cong B$ and $F([w_0]) \cong F$ as permutation groups. Suppose that $\chi_{[w_0]}$ is equal to χ .

Let $\Delta[w] := N_0([w])$ and $\Delta = \cup_{[w] \in R} \Delta[w]$. Let $\nu : \Delta \rightarrow R$ given in the obvious way by $\nu(\delta) = r$ if $\delta \in \Delta[w]$ and $[w] = r$. The group $\text{Aut}(N_0)$ acts on it and can be taken as automorphism group of Δ .

The fibres of ν form a partition of Δ invariant under the action of $\text{Aut}(N_0)$. Indeed, let $g \in \text{Aut}(R)$, consider $M^{-1}(g)$ which extends to $\bar{g} \in \text{Aut}(N_0)$. Then, if $\delta \in \Delta[w]$ there exists $n \in [w]$ such that $\delta \in N_0(n)$ and $\bar{g}\delta \in N_0(M^{-1}gw) \subseteq \Delta(g[w])$.

The fibre group at r_0 is equal to $F[w_0]$ and the binding group at r_0 is equal to $B([w_0])$. The map $\chi_{r_0} : \text{Aut}(R/r_0) \rightarrow F([w_0])/B([w_0])$ is exactly the composition of $M^{-1} : \text{Aut}(R/r_0) \rightarrow \text{Aut}(W/\{[w_0]\})$ and $\chi_{[w_0]}$. Since the data of π and ν are the same up to isomorphism, by Lemma 2.1.2 in [6] ν and π are isomorphic over R via the bijection $\beta(\delta) = ([w], \gamma(\hat{g}_r(\delta)))$, if $\delta \in \Delta[w]$ and $\hat{g}_r \in \text{Aut}(N_0)$ is an extension of $M^{-1}g_r$.

Let $\delta \in N_0(w)$ (so $\hat{g}_r\delta \in N_0(M^{-1}(g_r)w)$). Consider the bijection

$$\begin{array}{ccccccc} N_0 & \xrightarrow{id} & \Delta & \xrightarrow{\beta} & M & \xrightarrow{\alpha} & C_0 \\ \delta & \mapsto & \delta & \mapsto & ([w], \gamma(\hat{g}_r\delta)) & \mapsto & (w, \gamma(\hat{g}_r\delta)) \end{array}$$

Then $\alpha\beta \text{Aut}(N_0)\beta^{-1}\alpha^{-1} = \alpha \text{Aut}(M)\alpha^{-1} = \text{Aut}(C_0)$, i.e. $\text{Aut}(N_0)$ and $\text{Aut}(C_0)$ are isomorphic over W . \blacksquare

Example 27 Let W be a transitive structure, $w_0 \in W$, and ρ be an $\text{Aut}(W)$ -congruence on W . Assume that the permutation group induced by $\text{Aut}(W/\{[w_0]\})$ on $[w_0]$, which we shall denote by A , is closed in $\text{Sym}([w_0])$. Moreover suppose that the map M is injective, open and with closed image, as in Proposition 26. Let G be a finite permutation group acting on a set L . There always exists an almost-free finite cover. In order to see it, consider the wreath product $GWr_{[w_0]}A$ acting in the usual way on $[w_0] \times L$.

Let $\sigma : [w_0] \times L \rightarrow [w_0]$ given by $\sigma(w, l) = w$. Denote by B_1 the diagonal subgroup of $G^{[w_0]}$: it is normalized by A and so we can make the semidirect product $F_1 := B_1 \rtimes A$. This is closed by Proposition 9. Using the notation of Proposition 26 we have that χ is the homomorphism induced by restriction on $[w_0]$. Since χ is continuous, the hypothesis of Proposition 26 are satisfied and so we have an almost free finite cover $\pi : W \times L \rightarrow W$. We note that the automorphism group $\text{Aut}(W \times L)$, which we have got, is equal to $K_\rho \rtimes \text{Aut}(W)$ (using the notation of Theorem 11).

Now suppose that G is a simple non-abelian finite permutation group acting on itself by conjugation (so $G = L$). Next we give an example of an almost free finite cover with respect to ρ , not isomorphic to π , with kernel equal to $\ker \pi$.

Let $\pi : W \times G \rightarrow W$ be the cover that we have built above. Using the topological results in section 1.4 of [6] we have that the map $T : F_1 \rightarrow A$ is continuous, maps closed subgroups to closed subgroups and it is open. Then the isomorphism map $S : A \rightarrow F_1/B_1$ is a topological isomorphism.

Since $B([w_0]) = B_1 \cong G$, by conjugation of G by elements of $F([w_0]) = F_1$ we get a map $\gamma : F([w_0])/G \rightarrow \text{Out}(G)$. The image of γ is H/G , for some $H \leq \text{Aut}(G)$. Composing S with γ , we have a map

$$\bar{S} : A \rightarrow H/G.$$

In order to prove that γ is continuous we have to show that the kernel of γ is open. The kernel of γ is $C(G)_{F([w_0])}G/G$, where $C(G)_{F([w_0])}$ is the centralizer of G in $F([w_0])$. The group G is finite and hence closed in $F([w_0])$. Its orbits on $[w_0] \times G$ are finite and so it is also compact. Moreover, $C(G)_{F([w_0])}$ is closed. By Proposition 9 we have that $C(G)_{F([w_0])}G$ is closed in $F([w_0])$. Since it has finite index in $F([w_0])$, $C(G)_{F([w_0])}G$ is open in $F([w_0])$ and hence $C(G)_{F([w_0])}G/G$ is open in $F([w_0])/G$.

Let $P : H \rightarrow H/G$ be the quotient map and

$$F_2 := \{(\sigma, h) : \sigma \in A, h \in H \text{ and } P(h) = S(\sigma)\}$$

be the fibre product between A and H . This is a permutation group on $[w_0] \times G$ with action given by: $(\sigma, h)(w, g) = (\sigma w, h(g))$. By the same reasoning as in Proposition 23, we have that F_2 is closed in $\text{Sym}([w_0] \times G)$.

The group $B_2 := \{(id, g) : id \in \text{Sym}(\{[w_0]\}), g \in G\}$ is a normal subgroup of it. Let $\chi : \text{Aut}(W/\{[w_0]\}) \rightarrow F_2/B_2$ be the map given by

$$\chi(g) = (g|_{[w_0]}, h)B_2,$$

where h belongs to the coset $S(g_{|[w_0]})$. The map χ is well defined. Moreover, χ is continuous, since S is continuous.

Let

$$\sigma : [w_0] \times G \rightarrow [w_0]$$

be the projection on the first component. The induced map $F_2 \rightarrow \text{Sym}([w_0])$ has image A . Hence, by Proposition 26, we can build an almost-free finite cover π_0 w.r.t ρ with binding groups isomorphic to G . Note that the kernel is equal to K_ρ .

5.4 Problems

We described in an explicit way the kernels of expansions of the free finite cover of $\langle \Omega^{(n)}, \text{Sym}(\Omega) \rangle$, when the fibre groups and the binding groups are both isomorphic to a simple non-abelian finite group G .

1. What happens for finite covers where the base structure is a Grassmannian of a vector space over a finite field?
2. What happens for finite covers of $\Omega^{(n)}$ if the fibre groups and the binding groups are isomorphic to a simple abelian group? Here one would need to work with the closed $\text{Sym}(\Omega)$ -submodules of $\mathbb{F}_p^{\Omega^{(n)}}$. We remind that the case where the base permutation structure is $\langle [\Omega]^n, \text{Sym}(\Omega) \rangle$ was solved by Gray ([7]).

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